What's in a game?

Clovis Eberhart Tom Hirschowitz

Univ. Grenoble Alpes, Univ. Savoie Mont Blanc, CNRS, LAMA
Chambéry, France

July 8, 2018

Teasing question: what's your shortest proof that $\mathbb{1}$ is a topos?

Motivation

- Lots of game models.
- Lots of flavours.
- Common pattern.
- Subtle definitions and proofs.

Contribution

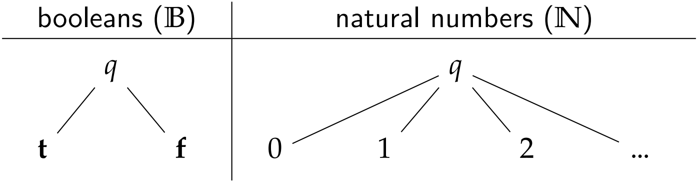
Abstract framework the construction of

- category of strategies
- subcategory of innocent strategies

in variants of HON, Tsukada-Ong, AJM.

HON Game Semantics: Games

A game Structures possible moves.



HON Game Semantics: Plays

Basically:

- sequence of moves
- interaction between program and environment

Example: `f = fun n -> 2 * n`

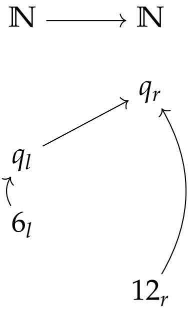
$$
\begin{array}{ccc}
\mathbb{N} & \longrightarrow & \mathbb{N} \\
 & & q_r \\
q_l & & \\
6_l & & \\
 & & 12_r
\end{array}
$$

HON Game Semantics: Plays

Basically:

- sequence of moves
- interaction between program and environment

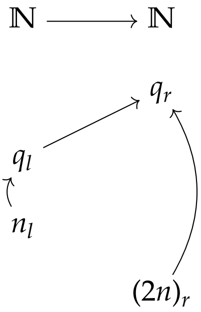
Example: `f = fun n -> 2 * n`



HON Game Semantics: Strategies

Strategy = prefix-closed set of accepted plays.

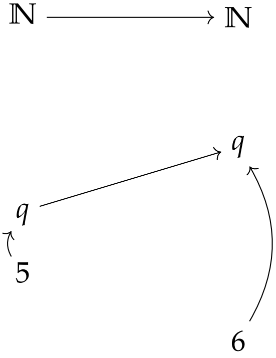
Example: for `f = fun n -> 2 * n`, (roughly) all plays of the form



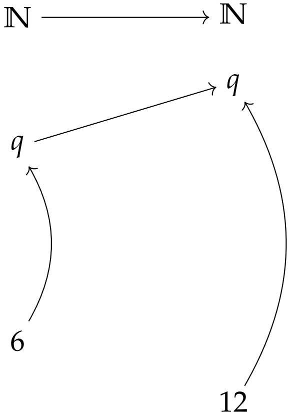
HON Game Semantics: Composition

Parallel composition + hiding.

```
f = fun n -> n + 1
```



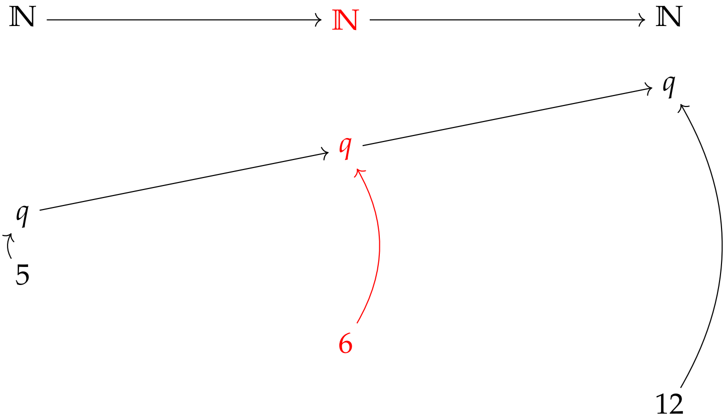

```
f = fun n -> 2 * n
```



HON Game Semantics: Composition

Parallel composition + hiding.

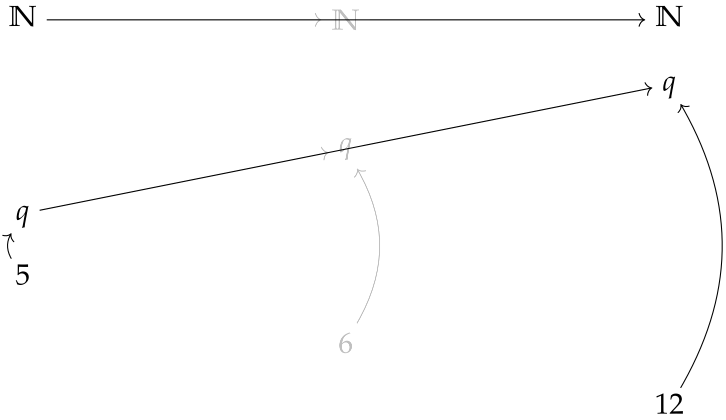
```
f = fun n -> n + 1          f = fun n -> 2 * n
```



HON Game Semantics: Composition

Parallel composition + hiding.

`f = fun n -> n + 1`   `f = fun n -> 2 * n`



Innocence

Idea: characterise purely functional programs.
Innocent strategy: can only change its behaviour based on its view.
View: certain type of play.

- strategy of counter:

$$q \quad 0 \qquad\qquad q \quad 1$$

- strategy of successor function:

$$q_r \quad q_l \quad 5_l \quad 6_r \qquad\qquad q_r \quad q_l \quad 5_l \quad 6_r$$

Innocence: the strategy accepts a play iff it accepts all its views.

Innocence

Idea: characterise purely functional programs.

Innocent strategy: can only change its behaviour based on its view.

View: certain type of play.

- strategy of counter:

$$q \quad 0 \qquad q \quad 1$$

- strategy of successor function:

$$q_r \quad q_l \quad 5_l \quad 6_r \qquad q_r \quad q_l \quad 5_l \quad 6_r$$

Innocence: the strategy accepts a play iff it accepts all its views.

Innocence

Idea: characterise purely functional programs.
Innocent strategy: can only change its behaviour based on its view.
View: certain type of play.

- strategy of counter:

$$q \quad 0 \qquad q \quad 1$$

- strategy of successor function:

$$q_r \quad q_l \quad 5_l \quad 6_r \qquad q_r \quad q_l \quad 5_l \quad 6_r$$

Innocence: the strategy accepts a play iff it accepts all its views.

All views

Innocence

Play p accepted iff all its views are.

- Let v be any view of play p.
- Only rarely witnessed by a morphism $v \to p$.

Idea (Melliès, then reworked by P. Levy)

Add new morphisms!

- Permutation equivalence $\sim$.
- Morphisms: $v \leqslant p' \sim p$ (proof relevant!).

The cone from views

First reformulation of innocence

Play p accepted iff v is for all morphisms $v \to p$.

New definition of strategies

Equivariance

New natural constraint: if $p \sim q$ then $(p \in \sigma) \Leftrightarrow q \in \sigma$.

Definition

Strategy = equivariant, prefix-closed set of plays.

New definition of strategies

Proposition

(Equivariant, prefix-closed sets of plays) $\simeq$ (Functors $\sigma\colon \mathit{Plays}^{op} \to \mathbb{2}$).

- $\mathbb{2} = (0 \to 1)$.
- $\sigma(p) = 1$ means accepted.
- Prefix-closedness: if q accepted, then

$$\sigma(p \leqslant q) = (\sigma(q) \leqslant \sigma(p)) = (1 \leqslant \sigma(p))$$

  hence p accepted.
- Equivariance: if p accepted, then

$$\sigma(p \sim q) = (\sigma(q) = \sigma(p)).$$

Nondeterminism for free!

- From functors $\mathbb{C}^{op} \to \mathbb{2}$, boolean presheaves,
- to functors $\mathbb{C}^{op} \to \mathrm{Set}$, i.e., presheaves $\widehat{\mathbb{C}}$.

Example

$$\begin{tikzcd} & \cdot \arrow[d, "a"] & \\ & \cdot \arrow[dl, "b"'] \arrow[dr, "c"] & \\ \cdot & & \cdot \end{tikzcd}
\qquad
\begin{tikzcd} & \cdot \arrow[dl, "a"'] \arrow[dr, "a"] & \\ \cdot \arrow[d, "b"'] & & \cdot \arrow[d, "c"] \\ \cdot & & \cdot \end{tikzcd}$$

$$\sigma(a) = \{x\} \qquad \sigma'(a) = \{x, x'\}$$

Corresponding functor to $\mathbb{2}$, as set of accepted plays:

$$|\sigma| = |\sigma'| = \{\varepsilon, a, ab, ac\}.$$

Innocence as a sheaf condition

Definition

$\sigma \in \widehat{\mathbb{P}_{A,B}}$ innocent iff sheaf for the topology induced by $\mathbb{V}_{A,B} \xhookrightarrow{\mathbf{i}_{A,B}} \mathbb{P}_{A,B}$.

Concretely:

$$
\begin{tikzcd}
v_1 \arrow[dr] & \dots & v_n \arrow[dl] \\
& p &
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
\sigma(v_1) & \dots & \sigma(v_n) \\
& \sigma(p). \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

- Boolean: $\sigma(p) = \sigma(v_1) \wedge \dots \wedge \sigma(v_n)$, ‘all views accepted’.
- Proof-relevant: compatible family of proofs of acceptance.

Proposition

σ innocent iff in essential image of $\prod_{\mathbf{i}_{A,B}}$.

Goal

- Associativity of proof-relevant composition (+ units).
- Stability of innocence under composition (+ units).

Abstracting away: recurring pattern

- Define games A, B, C, ...
- Define categories of plays $\mathbb{P}_{A,B}$.
- Define strategies $A \to B$ as prefix-closed sets of plays in $\mathbb{P}_{A,B}$.
- Composition $=$ parallel composition $+$ hiding.
- Identities $=$ copycat strategies.
- Prove that this defines a category of games and strategies.
- Define a notion of innocence.
- Prove that innocent strategies form a subcategory.

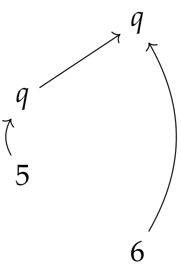
Abstracting away: recurring pattern

- Define games A, B, C, ... **Assumed!**
- Define categories of plays $\mathbb{P}_{A,B}$. **Assumed!**
- Define strategies $A \to B$ as prefix-closed sets of plays in $\mathbb{P}_{A,B}$.
- Composition $=$ parallel composition $+$ hiding.
- Identities $=$ copycat strategies.
- Prove that this defines a category of games and strategies.
- Define a notion of innocence. **Assumed!**
- Prove that innocent strategies form a subcategory.

Categories of plays

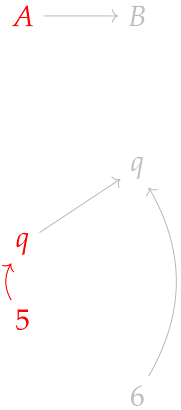
$$A \longrightarrow B$$

- Games A, B, C...
- Categories of plays $\mathbb{P}_A$, $\mathbb{P}_{A,B}$, $\mathbb{P}_{A,B,C}$...



Categories of plays

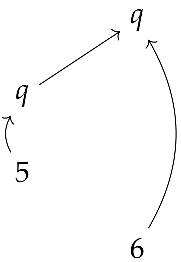
- Games A, B, C...
- Categories of plays $\mathbb{P}_A$, $\mathbb{P}_{A,B}$, $\mathbb{P}_{A,B,C}$...
- Projections $\mathbb{P}_{A,B} \to \mathbb{P}_A$.



Categories of plays

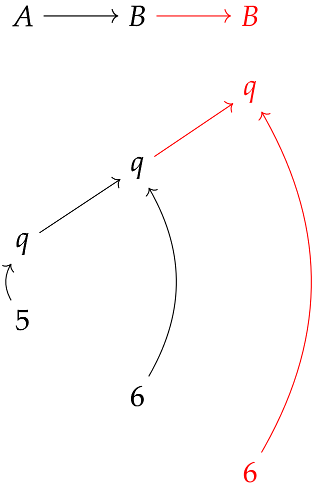
$$A \longrightarrow B$$

- Games A, B, C...
- Categories of plays $\mathbb{P}_A$, $\mathbb{P}_{A,B}$, $\mathbb{P}_{A,B,C}$...
- Projections $\mathbb{P}_{A,B} \to \mathbb{P}_A$.

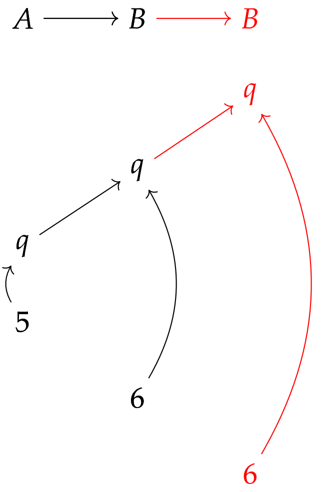


Categories of plays

- Games A, B, C...
- Categories of plays $\mathbb{P}_A$, $\mathbb{P}_{A,B}$, $\mathbb{P}_{A,B,C}$...
- Projections $\mathbb{P}_{A,B} \to \mathbb{P}_A$.
- Insertions $\mathbb{P}_{A,B} \to \mathbb{P}_{A,B,B}$.



Categories of plays



- Games A, B, C...
- Categories of plays $\mathbb{P}_A$, $\mathbb{P}_{A,B}$, $\mathbb{P}_{A,B,C}$...
- Projections $\mathbb{P}_{A,B} \to \mathbb{P}_A$.
- Insertions $\mathbb{P}_{A,B} \to \mathbb{P}_{A,B,B}$.
- Compatibility between projections and insertions.

Simplicial description of categories of plays

Definition

Game setting:

- set $\mathbb{A}$ of games,
- functor $\mathbb{P}\colon (\Delta/\mathbb{A})^{op} \to \mathrm{Cat}$.

where $\Delta/\mathbb{A}$ has as

- objects: lists $L = A_1, \ldots, A_n$ of games,
- morphisms:
  - coprojections $A, C \to A, B, C$,
  - coinsertions $A, A, B \to A, B$.

Strategies $A \to B$: $\widehat{\mathbb{P}_{A,B}}$.

Polynomial functors

If $F\colon \mathbb{C} \to \mathbb{D}$, let Δ_F denote restriction along F^{op}.
Well-known adjunction chain:

$$\begin{tikzcd} \widehat{\mathbb{C}} \arrow[rr, bend left, "\Sigma_F"] \arrow[rr, bend right, "\Pi_F"'] & \perp \\ \Delta_F \\ \perp & \widehat{\mathbb{D}}. \arrow[ll, "\Delta_F" description] \end{tikzcd}$$

Polynomial functor: composite of Δ's, $\prod$'s, and $\sum$'s.

Composition

Idea: parallel composition + hiding.

$$\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \xrightarrow{\Delta_{\delta_2+\delta_0}} \widehat{\mathbb{P}_{A,B,C} + \mathbb{P}_{A,B,C}} \xrightarrow{\Pi_{\nabla}} \widehat{\mathbb{P}_{A,B,C}} \dashrightarrow^{\Sigma_{\delta_1}} \widehat{\mathbb{P}_{A,C}}$$

Composition

Idea: parallel composition + hiding.

$$\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \xrightarrow{\Delta_{\delta_2+\delta_0}} \widehat{\mathbb{P}_{A,B,C} + \mathbb{P}_{A,B,C}} \xrightarrow{\Pi_\nabla} \widehat{\mathbb{P}_{A,B,C}} \dashrightarrow{\Sigma_{\delta_1}} \widehat{\mathbb{P}_{A,C}}$$

Composition

Idea: parallel composition + hiding.

$$\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \xrightarrow{\Delta_{\delta_2+\delta_0}} \widehat{\mathbb{P}_{A,B,C} + \mathbb{P}_{A,B,C}} \xrightarrow{\Pi_\nabla} \widehat{\mathbb{P}_{A,B,C}} \dashrightarrow{\Sigma_{\delta_1}} \widehat{\mathbb{P}_{A,C}}$$

Composition

Idea: parallel composition + hiding.

$$\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \xrightarrow{\Delta_{\delta_2+\delta_0}} \widehat{\mathbb{P}_{A,B,C} + \mathbb{P}_{A,B,C}} \xrightarrow{\Pi_\nabla} \widehat{\mathbb{P}_{A,B,C}} \dashrightarrow^{\Sigma_{\delta_1}} \widehat{\mathbb{P}_{A,C}}$$

Justification:

$\mathbf{m}_{A,B,C}(\sigma, \tau)$ accepts p

iff

iff

iff

Composition

Idea: parallel composition + hiding.

$$\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \xrightarrow{\Delta_{\delta_2+\delta_0}} \widehat{\mathbb{P}_{A,B,C} + \mathbb{P}_{A,B,C}} \xrightarrow{\Pi_\nabla} \widehat{\mathbb{P}_{A,B,C}} \dashrightarrow^{\Sigma_{\delta_1}} \widehat{\mathbb{P}_{A,C}}$$

Justification:

$$\begin{array}{ll} & \mathbf{m}_{A,B,C}(\sigma,\tau) \text{ accepts } p \\ \text{iff} & \text{there exists an interaction sequence } u \in \mathbb{P}_{A,B,C} \\ & \text{that is accepted and projects to } p \\ \text{iff} & \\ \text{iff} & \end{array}$$

Composition

Idea: parallel composition + hiding.

$$\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \xrightarrow{\Delta_{\delta_2+\delta_0}} \widehat{\mathbb{P}_{A,B,C} + \mathbb{P}_{A,B,C}} \xrightarrow{\Pi_\nabla} \widehat{\mathbb{P}_{A,B,C}} \dashrightarrow^{\Sigma_{\delta_1}} \widehat{\mathbb{P}_{A,C}}$$

Justification:

	$\mathbf{m}_{A,B,C}(\sigma,\tau)$ accepts p
iff	there exists an interaction sequence $u \in \mathbb{P}_{A,B,C}$ that is accepted and projects to p
iff	both $\mathrm{inl}\, u$ and $\mathrm{inr}\, u$ are accepted
iff	

Composition

Idea: parallel composition + hiding.

$$\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \xrightarrow{\Delta_{\delta_2+\delta_0}} \widehat{\mathbb{P}_{A,B,C} + \mathbb{P}_{A,B,C}} \xrightarrow{\Pi_\nabla} \widehat{\mathbb{P}_{A,B,C}} \dashrightarrow^{\Sigma_{\delta_1}} \widehat{\mathbb{P}_{A,C}}$$

Justification:

$$\begin{array}{ll} & \mathbf{m}_{A,B,C}(\sigma,\tau) \text{ accepts } p \\ \text{iff} & \text{there exists an interaction sequence } u \in \mathbb{P}_{A,B,C} \\ & \text{that is accepted and projects to } p \\ \text{iff} & \text{both } \mathrm{inl}\, u \text{ and } \mathrm{inr}\, u \text{ are accepted} \\ \text{iff} & \sigma \text{ accepts } \delta_2(u) \text{ and } \tau \text{ accepts } \delta_0(u). \end{array}$$

Composition

Idea: parallel composition + hiding.

$$\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \xrightarrow{\Delta_{\delta_2+\delta_0}} \widehat{\mathbb{P}_{A,B,C} + \mathbb{P}_{A,B,C}} \xrightarrow{\Pi_\nabla} \widehat{\mathbb{P}_{A,B,C}} \dashrightarrow^{\Sigma_{\delta_1}} \widehat{\mathbb{P}_{A,C}}$$

Justification:

$$\begin{array}{ll} & \mathbf{m}_{A,B,C}(\sigma,\tau) \text{ accepts } p \\ \text{iff} & \text{there exists an interaction sequence } u \in \mathbb{P}_{A,B,C} \\ & \text{that is accepted and projects to } p \\ \text{iff} & \text{both } \mathrm{inl}\, u \text{ and } \mathrm{inr}\, u \text{ are accepted} \\ \text{iff} & \sigma \text{ accepts } \delta_2(u) \text{ and } \tau \text{ accepts } \delta_0(u). \end{array}$$

Actually: proof-relevant version.

Copycat strategies

$$\mathbb{1} \cong \widehat{\varnothing} \xrightarrow{\Pi_!} \widehat{\mathbb{P}_A} \xrightarrow{\Sigma_{\iota_0}} \widehat{\mathbb{P}_{A,A}}$$

Justification:

$\propto_A$ accepts p

iff

iff

Copycat strategies

$$\mathbb{1} \cong \widehat{\varnothing} \xrightarrow{\Pi_!} \widehat{\mathbb{P}_A} \xrightarrow{\color{red}{\Sigma_{\iota_0}}} \widehat{\mathbb{P}_{A,A}}$$

Justification:

$\mathbb{c}_A$ accepts p

iff there exists a sequence $s \in \mathbb{P}_A$ that is accepted and mapped to p

iff

Copycat strategies

$$\mathbb{1} \cong \widehat{\emptyset} \xrightarrow{\Pi_!} \widehat{\mathbb{P}_A} \xrightarrow{\Sigma_{\iota_0}} \widehat{\mathbb{P}_{A,A}}$$

Justification:

$\quad\ \ \ \mathfrak{c}_A$ accepts p

iff there exists a sequence $s \in \mathbb{P}_A$ that is accepted and mapped to p

iff there is an s that is mapped to p.

Game settings

Additional condition for $\sum_{\delta_1}$ and $\sum_{\iota_0}$ to behave well

Projection $\delta_1 \colon \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and insertion $\iota_0 \colon \mathbb{P}_A \to \mathbb{P}_{A,A}$ should be discrete fibrations.

Associativity of composition

Theorem (Composition is associative):

$$\begin{tikzcd}
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C} + \mathbb{P}_{C,D}} \arrow[r, "\mathbf{m}_{A,B,C} + \mathbb{P}_{C,D}"] \arrow[d, "\mathbb{P}_{A,B} + \mathbf{m}_{B,C,D}"'] & \widehat{\mathbb{P}_{A,C} + \mathbb{P}_{C,D}} \arrow[d, "\mathbf{m}_{A,C,D}"] \\
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,D}} \arrow[r, "\mathbf{m}_{A,B,D}"'] & \widehat{\mathbb{P}_{A,D}}
\end{tikzcd}$$

commutes if

$$\begin{tikzcd}
\mathbb{P}_{A,B,C,D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,B,D} \arrow[d] \\
\mathbb{P}_{B,C,D} \arrow[r] & \mathbb{P}_{B,D}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbb{P}_{A,B,C,D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,C,D} \arrow[d] \\
\mathbb{P}_{A,B,C} \arrow[r] & \mathbb{P}_{A,C}
\end{tikzcd}$$

are pullbacks (*zipping lemma*).
Similar story for unitality of copycats.

Applications

Applications:

- HON,
- variants,
- AJM,
- TO.

May all be expressed as game settings, abstract composition agrees with traditional composition.

Innocent game settings

Add full subcategory views $\mathbb{V}_{A,B} \xrightarrow{\mathbf{i}_{A,B}} \mathbb{P}_{A,B}$ to the framework.

Definition

Innocent strategy: sheaf for the topology induced by $\mathbf{i}_{A,B}$.

Equivalently: in essential image of $\widehat{\mathbb{V}_{A,B}} \xrightarrow{\Pi_{\mathbf{i}_{A,B}}} \widehat{\mathbb{P}_{A,B}}$.

Stating preservation of innocence

The composite of any two innocent strategies is innocent.

(I.e., in the image of $\prod_{\mathbf{i}_{A,C}}$.)

$$\begin{tikzcd}
 & & \widehat{\mathbb{V}_{A,C}} \arrow[d, "\Pi_{\mathbf{i}_{A,C}}"] \\
\widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \arrow[r, "\Pi_{\mathbf{i}_{A,B}+\mathbf{i}_{B,C}}"'] \arrow[urr, dashed, bend left] & \widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \arrow[r, "\mathbf{m}_{A,B,C}"'] & \widehat{\mathbb{P}_{A,C}}
\end{tikzcd}$$

Proving preservation of innocence

Goal: composition of innocent strategies is again innocent.
Using alternative definition composition:

$$
\begin{tikzcd}
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[rr, "\Pi"] \arrow[d, equal] & & \mathbb{V}_{(A,B),(B,C)} \arrow[d, "\Pi"'] & \mathbb{V}_{A,B,C} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] \arrow[d, "\Pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{V}_{A,C} \arrow[d, "\Pi"] \\
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{A,B} + \mathbb{P}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"] \arrow[r, "\Sigma"'] & \mathbb{P}_{A,C}
\end{tikzcd}
$$

Proving preservation of innocence

Goal: composition of innocent strategies is again innocent.
Using alternative definition composition:

$$
\begin{tikzcd}
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[rr, "\Pi"] \arrow[d, equal] & & \mathbb{V}_{(A,B),(B,C)} \arrow[d, "\Pi"'] & \mathbb{V}_{A,B,C} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] \arrow[d, "\Pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{V}_{A,C} \arrow[d, "\Pi"] \\
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{A,B} + \mathbb{P}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"] \arrow[r, "\Sigma"'] & \mathbb{P}_{A,C}
\end{tikzcd}
$$

- Simple commutation.

Proving preservation of innocence

Goal: composition of innocent strategies is again innocent.
Using alternative definition composition:

$$\begin{tikzcd}
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[rr, "\Pi"] \arrow[d, equal] & & \mathbb{V}_{(A,B),(B,C)} \arrow[d, "\Pi"'] & \mathbb{V}_{A,B,C} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] \arrow[d, "\Pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{V}_{A,C} \arrow[d, "\Pi"] \\
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{A,B} + \mathbb{P}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"] \arrow[r, "\Sigma"'] & \mathbb{P}_{A,C}
\end{tikzcd}$$

- Simple commutation.
- Exact square (Guitart), modulo hypothesis (view-analyticity).

Proving preservation of innocence

Goal: composition of innocent strategies is again innocent.
Using alternative definition composition:

$$\begin{tikzcd}
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[rr, "\Pi"] \arrow[d, equal] & & \mathbb{V}_{(A,B),(B,C)} \arrow[d, "\Pi"'] & \mathbb{V}_{A,B,C} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] \arrow[d, "\Pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{V}_{A,C} \arrow[d, "\Pi"] \\
\mathbb{V}_{A,B} + \mathbb{V}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{A,B} + \mathbb{P}_{B,C} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,C)} & \mathbb{P}_{A,B,C} \arrow[l, "\Delta"] \arrow[r, "\Sigma"'] & \mathbb{P}_{A,C}
\end{tikzcd}$$

- Simple commutation.
- Exact square (Guitart), modulo hypothesis (view-analyticity).
- Distributive square, modulo hypothesis (locality).

Cf. distributive law in Harmer-Hyland-Melliès (LICS '07).

Local pushforward squares

Local pushforward square = pullback square

$$\begin{tikzcd} A \arrow[r, "V"] \arrow[d, "S"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "T"] \\ B \arrow[r, "U"'] & D \end{tikzcd}$$

with

1. U fully faithful,
2. T a discrete fibration,
3. and $T \cong \prod_U(S)$.

Or (3'): T corresponds to a sheaf for the topology induced by U.

Lemma

Local pushforward squares are distributive.

An important local pushforward square

$$\begin{tikzcd}
\mathbb{V}_{A,B,C} \arrow[r, "\mathbf{i}_{A,B,C}"] \arrow[d, "\delta_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\
\mathbb{V}_{A,C} \arrow[r, "\mathbf{i}_{A,C}"'] & \mathbb{P}_{A,C}
\end{tikzcd}$$

An important local pushforward square

$$
\begin{tikzcd}
\mathbb{V}_{A,B,C} \arrow[r, "\mathbf{i}_{A,B,C}"] \arrow[d, "\delta_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\
\mathbb{V}_{A,C} \arrow[r, "\mathbf{i}_{A,C}"'] & \mathbb{P}_{A,C}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & u_1 \arrow[dd, dashed] & \\
u_n \arrow[dd, dashed] & & \\
 & v_1 \arrow[dr, "f_1"] & \\
v_n \arrow[rr, "f_n"'] & & p
\end{tikzcd}
$$

An important local pushforward square

$$
\begin{tikzcd}
\mathbb{V}_{A,B,C} \arrow[r, "\mathbf{i}_{A,B,C}"] \arrow[d, "\delta_1"'] & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\
\mathbb{V}_{A,C} \arrow[r, "\mathbf{i}_{A,C}"'] & \mathbb{P}_{A,C}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & u_1 \arrow[dr, red] \arrow[dd, dashed] & \\
u_n \arrow[rr, red] \arrow[dd, dashed] & & \color{red}{u} \arrow[dd, dashed, red] \\
 & v_1 \arrow[dr, "f_1"] & \\
v_n \arrow[rr, "f_n"'] & & p
\end{tikzcd}
$$

- Locality hypothesis: δ_1 is a sheaf.
- Already noticed by Tsukada and Ong.

Summary

Contributions:

- Game settings: games, plays, interaction sequences, projections,...
- Abstract construction of categories of games and strategies.
- Subcategory of innocent strategies.
- Applications.

Not discussed here:

- Transfer between boolean and general presheaves.

Perspectives

- Structure on the category of strategies (monoidal, closed, etc).
- Generalisation (e.g., AJM saturated strategies).
- More exotic models (e.g., based on event structures).
- Categorification: bicategory of games and strategies.
- Game models $\rightsquigarrow$ game semantics.
- Better understand distributive squares.

Thank you.

And now, what's your shortest proof that $\mathbb{1}$ is a topos?

Result Transfer

Adjunction:

$$
\begin{tikzcd}
\mathrm{Set} \arrow[r, bend left, "\mathbf{l}"] \arrow[r, phantom, "\perp"] & 2 \arrow[l, bend left, "\mathbf{r}"]
\end{tikzcd}
$$

Result transfer:

$$
\begin{tikzcd}
\widehat{\mathbb{P}_{A,B}} \arrow[r, bend left, "\mathbf{l}_!"] \arrow[r, phantom, "\perp"] & \widetilde{\mathbb{P}_{A,B}} \arrow[l, bend left, "\mathbf{r}_!"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \arrow[r, "\mathbf{m}_{A,B,C}"] \arrow[d, "\mathbf{l}_!"'] & \widehat{\mathbb{P}_{A,C}} \arrow[d, "\mathbf{l}_!"] \\
\widetilde{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \arrow[r, "\overline{\mathbf{m}_{A,B,C}}"'] & \widetilde{\mathbb{P}_{A,C}}
\end{tikzcd}
$$

Exact Squares

$$\begin{tikzcd} \mathbb{A} \arrow[r, "T"] \arrow[d, "S"'] \arrow[dr, phantom, "\overset{\varphi}{\Longrightarrow}"] & \mathbb{C} \arrow[d, "V"] \\ \mathbb{B} \arrow[r, "U"'] & \mathbb{D} \end{tikzcd}$$

Mates:

$$\begin{tikzcd} \widehat{\mathbb{A}} \arrow[d, "\Sigma_S"'] \arrow[dr, phantom, "\Sigma_\varphi \Downarrow"] & \widehat{\mathbb{C}} \arrow[l, "\Delta_T"'] \arrow[d, "\Sigma_V"] \\ \widehat{\mathbb{B}} & \widehat{\mathbb{D}} \arrow[l, "\Delta_U"] \end{tikzcd} \qquad \begin{tikzcd} \widehat{\mathbb{A}} \arrow[r, "\prod_T"] \arrow[dr, phantom, "\overset{\prod_\varphi}{\Longleftarrow}"] & \widehat{\mathbb{C}} \\ \widehat{\mathbb{B}} \arrow[u, "\Delta_S"] \arrow[r, "\prod_U"'] & \widehat{\mathbb{D}} \arrow[u, "\Delta_V"'] \end{tikzcd}$$

Exact square: $\sum_\varphi$ (equivalently $\prod_\varphi$) is an isomorphism.
Guitart: conditions for square to be exact.

Distributive Squares

Conditions for

$$
\begin{tikzcd}
\widehat{\mathbb{A}} \arrow[d, "\Sigma_S"'] & \widehat{\mathbb{C}} \arrow[l, "\Pi_T"'] \arrow[d, "\Sigma_V"] \\
\widehat{\mathbb{B}} & \widehat{\mathbb{D}} \arrow[l, "\Pi_U"]
\arrow[from=1-1, to=2-2, phantom, "\overset{\tilde{\varphi}}{\Longrightarrow}"]
\end{tikzcd}
$$

to commute.

Composition

Associativity of composition:

$$\begin{tikzcd}
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C} + \mathbb{P}_{C,D}} \arrow[r, "\mathbf{m}_{A,B,C} + \mathbb{P}_{C,D}"] \arrow[d, "\mathbb{P}_{A,B} + \mathbf{m}_{B,C,D}"'] & \widehat{\mathbb{P}_{A,C} + \mathbb{P}_{C,D}} \arrow[d, "\mathbf{m}_{A,C,D}"] \\
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,D}} \arrow[r, "\mathbf{m}_{A,B,D}"'] & \widehat{\mathbb{P}_{A,D}}
\end{tikzcd}$$

Associativity of Composition

$$
\begin{tikzcd}
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C} + \mathbb{P}_{C,D}} \arrow[r, "\Pi"] \arrow[d, "\Pi"'] & \widehat{\mathbb{P}_{(A,B),(B,C)} + \mathbb{P}_{C,D}} \arrow[r, "\Delta"] & \widehat{\mathbb{P}_{A,B,C} + \mathbb{P}_{C,D}} \arrow[r, "\Sigma"] & \widehat{\mathbb{P}_{A,C} + \mathbb{P}_{C,D}} \arrow[d, "\Pi"] \\
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{(B,C),(C,D)}} \arrow[d, "\Delta"'] & & & \widehat{\mathbb{P}_{(A,C),(C,D)}} \arrow[d, "\Delta"] \\
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C,D}} \arrow[d, "\Sigma"'] & & & \widehat{\mathbb{P}_{A,C,D}} \arrow[d, "\Sigma"] \\
\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,D}} \arrow[r, "\Pi"'] & \widehat{\mathbb{P}_{(A,B),(B,D)}} \arrow[r, "\Delta"'] & \widehat{\mathbb{P}_{A,B,D}} \arrow[r, "\Sigma"'] & \widehat{\mathbb{P}_{A,D}}
\end{tikzcd}
$$

Proof: Associativity of Composition

$$
\begin{tikzcd}
\mathbb{P}_{A,B} + \mathbb{P}_{B,C} + \mathbb{P}_{C,D} \arrow[r, "\Pi"] \arrow[d, "\Pi"'] & \mathbb{P}_{(A,B),(B,C)} + \mathbb{P}_{C,D} & \mathbb{P}_{A,B,C} + \mathbb{P}_{C,D} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] & \mathbb{P}_{A,C} + \mathbb{P}_{C,D} \arrow[d, "\Pi"] \\
\mathbb{P}_{A,B} + \mathbb{P}_{(B,C),(C,D)} & & & \mathbb{P}_{(A,C),(C,D)} \\
\mathbb{P}_{A,B} + \mathbb{P}_{B,C,D} \arrow[u, "\Delta"] \arrow[d, "\Sigma"'] & & & \mathbb{P}_{A,C,D} \arrow[u, "\Delta"'] \arrow[d, "\Sigma"] \\
\mathbb{P}_{A,B} + \mathbb{P}_{B,D} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,D)} & \mathbb{P}_{A,B,D} \arrow[l, "\Delta"] \arrow[r, "\Sigma"'] & \mathbb{P}_{A,D}
\end{tikzcd}
$$

Proof: Associativity of Composition

$$
\begin{tikzcd}
\mathbb{P}_{A,B} + \mathbb{P}_{B,C} + \mathbb{P}_{C,D} \arrow[r, "\Pi"] \arrow[d, "\Pi"'] & \mathbb{P}_{(A,B),(B,C)} + \mathbb{P}_{C,D} \arrow[d, "\Pi"] & \mathbb{P}_{A,B,C} + \mathbb{P}_{C,D} \arrow[l, "\Delta"'] \arrow[r, "\Sigma"] & \mathbb{P}_{A,C} + \mathbb{P}_{C,D} \arrow[d, "\Pi"] \\
\mathbb{P}_{A,B} + \mathbb{P}_{(B,C),(C,D)} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,C),(C,D)} & & \mathbb{P}_{(A,C),(C,D)} \\
\mathbb{P}_{A,B} + \mathbb{P}_{B,C,D} \arrow[u, "\Delta"] \arrow[d, "\Sigma"'] & & \mathbb{P}_{A,B,C,D} \arrow[ul, "\Delta"'] \arrow[r, "\Sigma"] \arrow[d, "\Sigma"'] & \mathbb{P}_{A,C,D} \arrow[u, "\Delta"'] \arrow[d, "\Sigma"] \\
\mathbb{P}_{A,B} + \mathbb{P}_{B,D} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,D)} & \mathbb{P}_{A,B,D} \arrow[l, "\Delta"] \arrow[r, "\Sigma"'] & \mathbb{P}_{A,D}
\end{tikzcd}
$$

Proof: Associativity of Composition (cont.)

$$
\begin{tikzcd}
\mathbb{P}_{A,B} + \mathbb{P}_{(B,C),(C,D)} \arrow[rr, "\Pi"] & & \mathbb{P}_{(A,B),(B,C),(C,D)} \\
\mathbb{P}_{A,B} + \mathbb{P}_{B,C,D} \arrow[u, "\Delta"] \arrow[r, "\Pi"] \arrow[dr, "\Pi"'] \arrow[dd, "\Sigma"] & \mathbb{P}_{(A,B),(B,C),(C,D)} & \mathbb{P}_{A,B,C,D} \arrow[u, "\Delta"'] \arrow[l, "\Delta"'] \arrow[dl, "\Delta"] \arrow[dd, "\Sigma"'] \\
& \mathbb{P}_{(A,B),(B,C,D)} \arrow[u, "\Delta"] \arrow[d, "\Sigma"] & \\
\mathbb{P}_{A,B} + \mathbb{P}_{B,D} \arrow[r, "\Pi"'] & \mathbb{P}_{(A,B),(B,D)} & \mathbb{P}_{A,B,D} \arrow[l, "\Delta"]
\end{tikzcd}
$$

Applications

Applications: HON, variants, AJM, TO.
May all be expressed as game settings, abstract composition agrees with traditional composition.
Subtleties:

- HON: liberal definition of $\mathbb{P}_A$ (for projections $\mathbb{P}_{A,B} \to \mathbb{P}_A$ to exist)
- AJM: slightly different definition of $\mathbb{P}_{A,B,C}$ (projection $\mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ should be a discrete fibration)

Blass games: composition known to be non-associative. Cannot be expressed as a game setting (zipping fails).

Conditions to Preserve Innocence

Locality: $\delta_1 \colon \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and $\iota_0 \colon \mathbb{P}_A \to \mathbb{P}_{A,A}$ are sheaves.

$$\begin{tikzcd} & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\ \mathbb{V}_{A,C} \arrow[r, "\mathbf{i}_{A,C}"'] & \mathbb{P}_{A,C} \end{tikzcd}$$

View-analyticity:

Conditions to Preserve Innocence

Locality: $\delta_1 \colon \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and $\iota_0 \colon \mathbb{P}_A \to \mathbb{P}_{A,A}$ are sheaves.

$$\begin{tikzcd}
\mathbb{V}_{A,B,C} \arrow[r, "\mathbf{i}_{A,B,C}"] \arrow[d, "\delta_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\
\mathbb{V}_{A,C} \arrow[r, "\mathbf{i}_{A,C}"'] & \mathbb{P}_{A,C}
\end{tikzcd}$$

View-analyticity:

Conditions to Preserve Innocence

Locality: $\delta_1 \colon \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and $\iota_0 \colon \mathbb{P}_A \to \mathbb{P}_{A,A}$ are sheaves.

$$
\begin{tikzcd}
\mathbb{V}_{A,B,C} \arrow[r, "\mathbf{i}_{A,B,C}"] \arrow[d, "\delta_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\
\mathbb{V}_{A,C} \arrow[r, "\mathbf{i}_{A,C}"'] & \mathbb{P}_{A,C}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & u_1 \arrow[dd, dashed] & \\
u_n \arrow[dd, dashed] & & \\
 & v_1 \arrow[dr, "f_1"] & \\
v_n \arrow[rr, "f_n"'] & & p
\end{tikzcd}
$$

View-analyticity:

Conditions to Preserve Innocence

Locality: $\delta_1 \colon \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and $\iota_0 \colon \mathbb{P}_A \to \mathbb{P}_{A,A}$ are sheaves.

$$\begin{tikzcd}
\mathbb{V}_{A,B,C} \arrow[r, "\mathbf{i}_{A,B,C}"] \arrow[d, "\delta_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\
\mathbb{V}_{A,C} \arrow[r, "\mathbf{i}_{A,C}"'] & \mathbb{P}_{A,C}
\end{tikzcd}
\qquad
\begin{tikzcd}
& u_1 \arrow[dr, red] \arrow[dd, dashed] & \\
u_n \arrow[rr, red] \arrow[dd, dashed] & & \color{red}{u} \arrow[dd, dashed, red] \\
& v_1 \arrow[dr, "f_1"] & \\
v_n \arrow[rr, "f_n"'] & & p
\end{tikzcd}$$

View-analyticity:

Conditions to Preserve Innocence

Locality: $\delta_1 \colon \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and $\iota_0 \colon \mathbb{P}_A \to \mathbb{P}_{A,A}$ are sheaves.

$$\begin{tikzcd}
\mathbb{V}_{A,B,C} \arrow[r, "\mathbf{i}_{A,B,C}"] \arrow[d, "\delta_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\
\mathbb{V}_{A,C} \arrow[r, "\mathbf{i}_{A,C}"'] & \mathbb{P}_{A,C}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & u_1 \arrow[dr, red] \arrow[dd, dashed] & \\
u_n \arrow[rr, red] \arrow[dd, dashed] & & \color{red}{u} \arrow[dd, dashed, red] \\
 & v_1 \arrow[dr, "f_1"] & \\
v_n \arrow[rr, "f_n"'] & & p
\end{tikzcd}$$

View-analyticity:

$$\mathbf{i}_{A,B}(v) \xrightarrow{\quad f \quad} \delta_2(u)$$

Conditions to Preserve Innocence

Locality: $\delta_1 \colon \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and $\iota_0 \colon \mathbb{P}_A \to \mathbb{P}_{A,A}$ are sheaves.

$$\begin{tikzcd}
\mathbb{V}_{A,B,C} \arrow[r, "\mathbf{i}_{A,B,C}"] \arrow[d, "\delta_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{A,B,C} \arrow[d, "\delta_1"] \\
\mathbb{V}_{A,C} \arrow[r, "\mathbf{i}_{A,C}"'] & \mathbb{P}_{A,C}
\end{tikzcd}$$

$$\begin{tikzcd}
& u_1 \arrow[dr, red] \arrow[dd, dashed] & \\
u_n \arrow[rr, red] \arrow[dd, dashed] & & \color{red}{u} \arrow[dd, dashed, red] \\
& v_1 \arrow[dr, "f_1"] & \\
v_n \arrow[rr, "f_n"'] & & p
\end{tikzcd}$$

View-analyticity:

$$\begin{tikzcd}
\mathbf{i}_{A,B}(v) \arrow[rr, "f"] \arrow[dr, "g"'] & & \delta_2(u) \\
& \delta_2(\mathbf{i}_{A,B,C}(w)) \arrow[ur, "\delta_2(h)"'] &
\end{tikzcd}$$

Boolean Innocent Strategies

But (non-deterministic) innocent strategies should not compose!
Answer:

$$\begin{tikzcd}
\widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \arrow[r] & \widehat{\mathbb{P}_{A,C}} \arrow[d, "\mathbf{l}_!"] \\
\widetilde{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \arrow[u, "\mathbf{r}_!"] \arrow[r] & \widetilde{\mathbb{P}_{A,C}}
\end{tikzcd}$$

does not commute.

- *concurrent* innocent strategies compose
- traditional innocent strategies do not