Algebraic structures from shapes

Tom Hirschowitz

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LAMA
Laboratoire de Mathématiques
Université de Savoie

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Introduction

- Mainly an introduction to the theory of nerves (Berger, Leinster, Weber, . . .).
- Why?

Why nerves?

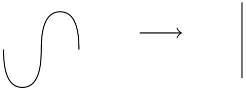
Some of us design algebraic structures modelling computation.

- Categorical semantics (Lambek-Scott, Seely, Lafont, Benton-Hyland-Bierman, . . .).
- Polygraphic approach (Burroni, Lafont, . . .).

Some design graphical structures modelling computation.

- Proof nets, interaction nets (Girard, Lafont, Mazza).
- Polygraphic approach (Burroni, Lafont, . . .).

Polygraphs not so graphical: need a rewrite



Algebraic vs. graphical

- Algebraic approaches define categories of models.
- Most graphical approaches define one model.

Graphical seems weaker.

Nerve theory

Graphical $\rightsquigarrow$ algebraic (in good cases).

Specifically:

- Given a nice monad T,
- compute a sketch S_T,
- such that

T-algebras $\simeq$ Models of S_T.

Why nerves?

If T is nice:

T-algebras $\simeq$ Models of S_T.

- Here: nice means local right adjoint.
- Has to do with being graphical.

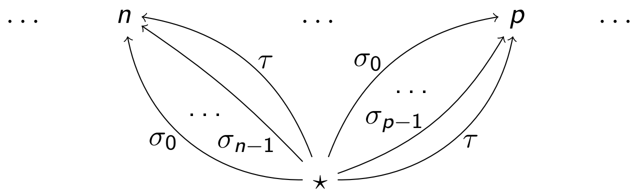
Hope: this can be useful in computer science, and in Geocal in particular.

Contents

- One example: (symmetric) multicategories (Lambek?).
- One counterexample: 2-categories.

Shapes

Consider the category $\mathcal{S}$ looking like:

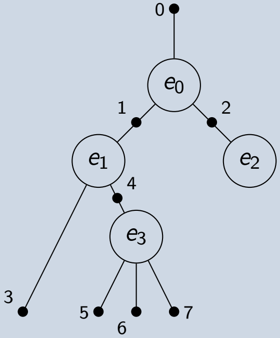


Presheaves over shapes

The category $\hat{\mathcal{S}} = [\mathcal{S}^{op}, \mathrm{Set}]$ has multigraphs as objects.

Example

- $F(\star) = \{0, \ldots, 7\}$,
- $F(0) = \{e_2\}$,
- $F(2) = \{e_0, e_1\}$,
- $F(3) = \{e_3\}$,
- $F(\tau)(e_0) = 0$,
- $F(\tau)(e_1) = 1$,
- $F(\sigma_0)(e_0) = 1, \ldots$



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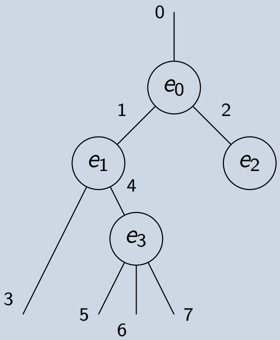


When every vertex is at most 1-in 1-out: forget the bullets!

Multicategories

There is a "free multicategory" monad $\mathcal{M}$ on $\hat{\mathcal{S}}$, which we will reconstruct graphically.

We now:

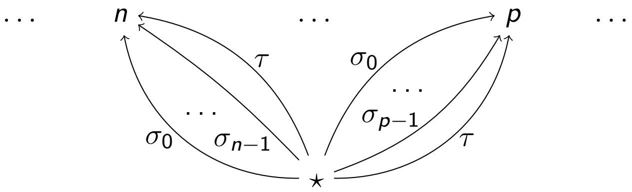
- define a recipient presheaf $\mathcal{R}$ on $\mathcal{S}$;
- and a sequence of subpresheaves $\mathcal{T}_0, \mathcal{T}_1, \ldots \subseteq \mathcal{R}$;
- consider the union $\mathcal{T}_\omega = \bigcup_{i \in \omega} \mathcal{T}_i \subseteq \mathcal{R}$.

Then:

- $\mathcal{T}_\omega(n) \cong$ morphisms $n \to 1$ in the free multicategory over 1;
- And we reconstruct $\mathcal{M}$ from this $\mathcal{T}_\omega$.

The recipient presheaf

- Recall $\mathcal{S}$:



and observe that $\mathcal{S}/\star \cong 1$.

The recipient presheaf

- Let $\mathcal{R}(\star)$ have as sole element the representable $y\star$, seen as a functor $\mathcal{S}/\star \to \hat{\mathcal{S}}$;
- $\mathcal{R}(n)$ is the set of functors $\mathcal{S}/n \to \hat{\mathcal{S}}$ of the shape:

$$
\begin{array}{ccccc}
 & & \star & & \\
 & & t \downarrow & & \\
 & & f & & \\
 & s_0 \nearrow & & \nwarrow s_{n-1} & \\
\star & & \ldots & & \star
\end{array}
$$

with f finite (let us call them multi-cospans),
- modulo isomorphism of multi-cospans $\mathcal{S}/n \to \hat{\mathcal{S}}$.

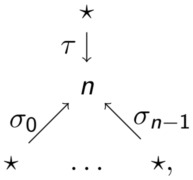
Intuition

Multigraphs with arity and handles.

Boot

Consider the following presheaf $\mathcal{T}_0 \subseteq \mathcal{R}$:

- $\mathcal{T}_0(\star) = \mathcal{R}(\star) = \{\star\}$,
- $\mathcal{T}_0(n)$ is the singleton

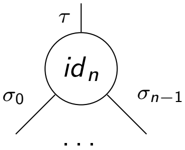


plus

(when $n = 1$).

Boot

Pictorially, $\mathcal{T}_0$ has, for each n, one element



plus

(when $n = 1$).

Role of the diagram: distinguish the dangling wires.

Step

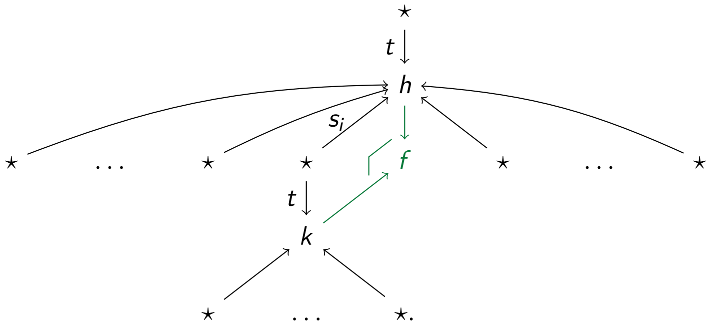
Now define $\mathcal{T}_{n+1}$ to be the union of $\mathcal{T}_n$ and T'_n, which has:

- $T'_n(\star) = \emptyset$

Step

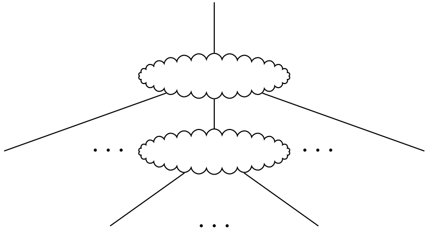
Now define $\mathcal{T}_{n+1}$ to be the union of $\mathcal{T}_n$ and T'_n, which has:

- $T'_n(m)$ is the set of diagrams $(f, \bar{s}, t)$ as above such that:
  - there exist $p + q - 1 = m$, $i \in p$, and
  - diagrams $h \in \mathcal{T}_n(p)$ and $k \in \mathcal{T}_n(q)$, such that f is:



Picture

This just glues two multigraphs together along the chosen edge:



Wrap up

Definition

Let $\mathcal{T}_\omega$ be the union of all the $\mathcal{T}_n$s.

Result

Theorem (Coherence at 1)

For all n, $\mathcal{T}_\omega(n)$ is isomorphic to the set $M(n)$ of morphisms $n \longrightarrow 1$ in the free multicategory on 1.
Furthermore, composition and identities are given by the operations on the $\mathcal{T}_n$'s, e.g,

$$\begin{array}{ccc} \mathcal{T}_\omega(p) \times \mathcal{T}_\omega(q) & \cong & M(p) \times M(q) \\ {\scriptstyle \text{glueing at } i}\downarrow & & \downarrow{\scriptstyle \circ_i} \\ \mathcal{T}_\omega(p+q-1) & \cong & M(p+q-1) \end{array}$$

commutes.

The monad

We now derive the monad from the presheaf $\mathcal{T}_\omega$.

Consider the functor:

$$\begin{array}{ccc} \hat{\mathcal{S}} & \longrightarrow & \hat{\mathcal{S}} \\ F & \mapsto & \mathcal{T}F, \end{array}$$

where $\mathcal{T}(F)(s) = \coprod_{x \in \mathcal{T}_\omega(s)} \hat{\mathcal{S}}(x(id_s), F)$.

WTF?

Sorry, I have to show you this key formula.

Understanding the key formula

$$\mathcal{T}(F)(s) = \coprod_{x \in \mathcal{T}_\omega(s)} \hat{\mathcal{S}}(x(id_s), F)$$

- Recall that $\mathcal{T}_\omega(s)$ is a set of diagrams $\mathcal{S}/s \to \hat{\mathcal{S}}$, e.g.,

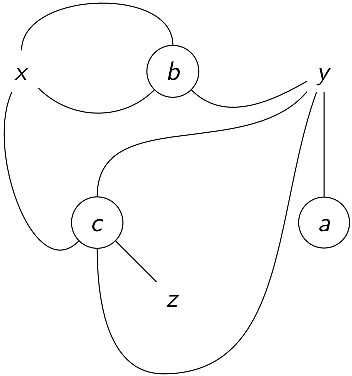
$$\begin{array}{ccc} & \star & \\ & t \downarrow & \\ & f & \\ s_0 \nearrow & & \nwarrow s_{n-1} \\ \star & \ldots & \star, \end{array}$$

- So that for $x \in \mathcal{T}_\omega(s)$,
  - $x(id_s)$ is a presheaf on $\mathcal{S}$,
  - here f.
  - And a natural transformation $f \to F$ is a labelling of f in F.

Example

Let:

- $F(\star) = \{x, y, z\}$,
- $F(0) = \{a\}$,
- $F(2) = \{b\}$,
- $F(3) = \{c\}$,
- $F(\tau)(a) = y$,
- $F(\tau)(b) = x$,
- $F(\sigma_0)(b) = x, \ldots$

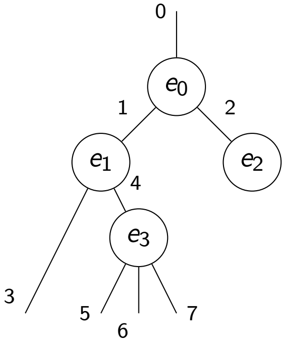


Seen as specifying operations

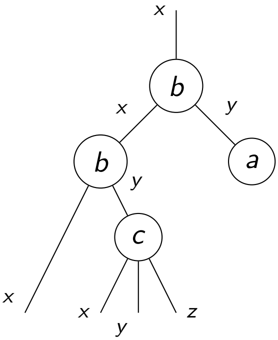
$$a \colon I \to y \qquad b \colon x \otimes y \to x \qquad c \colon x \otimes y \otimes z \to y.$$

Example

Then remember f :



A natural transformation $f \longrightarrow F$ sends each element to F, consistently.



A last time

The key formula again:

$$\mathcal{T}(F)(s) = \coprod_{x \in \mathcal{T}_\omega(s)} \hat{\mathcal{S}}(x(id_s), F),$$

i.e.,

$$\mathcal{T}(F)(s) = \coprod_{(f,\ldots) \in \mathcal{T}_\omega(s)} \hat{\mathcal{S}}(f, F).$$

Results

Theorem (Not me)

- *$\mathcal{T}$ is a lra monad, and a club [Kelly,Leinster,Weber]:*
  - *$\mathcal{T}$ preserves pullbacks.*
  - *Naturality squares for μ and η are pullbacks.*
  - *Generic factorisations.*
  - *$\mathcal{T}$ is sketchable, i.e., algebraic.*
- *$\mathcal{T}$ is isomorphic to the "free multicategory" monad $\mathcal{M}$.*

Symmetric multicategories

In passing: the technique trivially extends to symmetric multicategories.

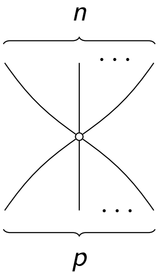
Key observation

The involved functors $\mathcal{S}/n \to \hat{\mathcal{S}}$ (the multi-cospans) have no automorphisms.

(I.e., no non-trivial endo-isomorphisms.)
Let us now illustrate why this is key.

A new base category

We may define a new category $\mathcal{S}_2$ whose representables look like



Operations

- Horizontal composition: glueing along backgrounds.
- Vertical composition: glueing along wires.

Argh

- One defines $\mathcal{T}_\omega$ using these operations and identities,
- then $\mathcal{T}$ using the key formula

$$\mathcal{T}(F)(s) = \coprod_{x \in \mathcal{T}_\omega(s)} \hat{\mathcal{S}}(x(id_s), F),$$

with:

Hope

$\mathcal{T} \cong$ "free 2-category".

But this is wrong, because of the 0-ary case.
Very similar to an old error by Carboni and Johnstone.

Observation

Our diagrams do have automorphisms:

$$\boxed{x \qquad y} \qquad \cong \qquad \boxed{y \qquad x}$$

Think of two 2-cells $id \to id$.

Consequences of our observation

- Consider the presheaf F on $\mathcal{S}_2$ with one object, no morphism, and exactly two 2-cells 0 and 1.
- And the two labellings of the above $\boxed{\mathrm{x} \quad \mathrm{y}}$ in $\{0, 1\}$:
  - $x \mapsto 0, y \mapsto 1$, and
  - $x \mapsto 1, y \mapsto 0$.
- Since $\boxed{\mathrm{x} \quad \mathrm{y}} \cong \boxed{\mathrm{y} \quad \mathrm{x}}$, they count as two labellings of the same element of $\mathcal{T}(0, 0)$.

Problem

By the Eckmann-Hilton argument they are equal in the free 2-category on F.

Conclusion

- A technique to define algebraic structures from graphical ones.
- Problems when the graphs have automorphisms.

Room for improvement:

- Leinster-Weber provide too few tools to prove that $\mathcal{T}$ is a monad (even with conditions on $\mathcal{T}_\omega$),
- Better handle graphs with automorphisms (perhaps weaken the theory by not quotienting under isomorphism).