Towards a theory of programming languages

Tom Hirschowitz

LAMA, CNRS et Université de Savoie

The ladder of abstraction (© Brett Victor): level 1

Formal reasoning over practical things like

counting:	integers,
moving:	transformations of the plane,
exchanging:	permutations.

The ladder of abstraction: level 2

Common properties of:

- integers,
- rotations,
- translations,
- symmetries,
- isometric,transformations.
- permutations on a finite set...

Abstraction

Notion of group.

The ladder of abstraction: towards level 3

Neighbours

- Generalisation: monoids (e.g., all transformations of the plane).
- Specialisation: rings, modules, algebras, ...

Problem

- Numerous abstract notions.
- A lot in common:
  - free constructions (e.g., free monoid = words);
  - notions of morphisms;
  - downcasting (any ring is a group, at least).

The ladder of abstraction: level 3

Abstraction over abstraction!

Two proposals:

- Lawvere theories;
- monads on sets.

Lawvere theories a little more general.

Programming language theory: still at level 2!

Level 1: calculi

Programming languages $\rightsquigarrow$ calculi:

- pure λ-calculus,
- λ-calculus in cbv, cbn, lazy, fully lazy, optimal, ...
- λ-calculus with `let rec` / refs / `call/cc`, ...
- ζ, λσ, $\bar{\lambda}\mu\tilde{\mu}$, ...
- π, join, Ambients, Spy, ...

Analogy

Reasoning over integers ≈ Reasoning over programs in language L.

And that's only the untyped tip of the iceberg!

Level 2: denotational semantics

- Denotational model of language L $\approx$ mathematical structure supporting the operations of L.
- L $\approx$ `free' structure.

Common use of denotational semantics

Disprove a property of programs by finding a counter-model.

Groups:	integers,permutations, ...
Rings:	integers, polynomials, ...
Models of simply-typed λ:	cartesian closed categories, ...

Rk: not all calculi are at level 2 (i.e., equipped with a notion of model).

Towards level 3?

Groups:	integers,permutations, …	Lawvere theories
Rings:	integers, polynomials, …	
X-calculus:	models of X,	Lawvere theory?

No, for at least two reasons:

- variable binding ($\lambda x\,.\,x \equiv \lambda x'.\,x'$);
- dynamics $(\lambda x\,.\,M)N \rightsquigarrow M[x \mapsto N]$: need for `directed' equations.

Need to generalise Lawvere theories!

Hence the question

- What is a programming language?
- What is a translation between two programming languages?
- General results?

Let us start with a (fake) poll...

Fake poll: what is a programming language?

Low-level answer

- a language on a finite alphabet,
- a translation to `x86` (...).

- Why definitely fix `x86`?
  - Low-level does evolve a lot (shit, it's `amd64` already...).
  - Sometimes overkill, e.g., language of regular expressions.
  - Not canonical.
  - Limited: e.g., distributed computing.
- Not won at the level of high-level reasoning on programs.

Fake poll: what is a programming language?

Frequent answer by researchers in programming languages

One super complex language supposed to model all other languages.

- Illusory.
- Adding features may change global properties: study of fragments.
- Eludes the crucial question of what morphisms should be.

Fake poll: what is a programming language?

Other frequent answer mostly in the UK

A structural operational semantics (Plotkin, 1981), in a certain format.

- Low-level notion of syntax with binding.
- Morphisms: only starting to be investigated.
- Far from mainstream mathematics.

Fake poll: what is a programming language?

Other frequent answer

A higher-order rewrite system (Nipkow, 1991).

Close answer: a combinatory reduction system (Klop, 1980).

- Roughly, rewriting terms with variable binding.
- No notion of morphism, even google does not find anything.
- Far from mainstream mathematics.

About the last two,

- structural operational semantics (SOS) and
- higher-order rewriting (HOR).

- General results, but on one language:
  - congruence of bisimilarity (SOS),
  - confluence, finite developments, etc (HOR).
- Better, hints at level 3:
  - mathematical operational semantics (SOS, Turi and Plotkin),
  - cartesian closed 2-categories (HOR, main subject here).

Lawvere theories in 20 slides

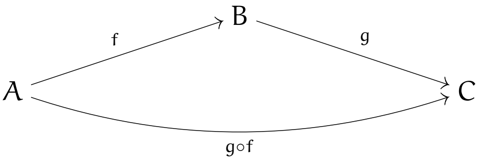
Starters: introduction to category theory, then Lawvere theories.

Categories

Definition

Category: a (directed, multi) graph equipped with

- a composition law on edges,
- identities.

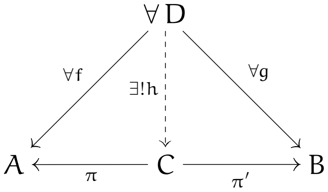


Examples

- The category Grp.
  - Vertices / objects: groups.
  - Edges / morphisms: morphisms of groups.
- Large category.
- Similar examples: monoids, rings, etc.
- Topological spaces and continuous functions: Top.
- Graphs: Gph.
- Even plain sets: Set.

Cartesian product (in any category!)

- Consider any objects $A, B \in \mathcal{C}$;
- $A \xleftarrow{\pi} C \xrightarrow{\pi'} B$ is a product of A and B iff



- Notation: $C = A \times B$ and $h = \langle f, g \rangle$.

Example

- Set, Grp, Top,...
- Graphs.

Terminal object

- A is a terminal object in $\mathcal{C}$ iff $\forall B \overset{\exists! f}{\dashrightarrow} A$.
- Notation: $A = 1$, $f = !B$.

Example

- Sets: singleton.
- Graphs: what would you guess?

Finite products = products + terminal objects

Definition

Category with finite products:

- a product $(A \times B, \pi, \pi')$ for all A, B,
- a terminal object 1.

First insight of Lawvere theories

Observation

Any model of an algebraic theory `is' a category with finite products.

I.e.,

- Any monoid is a category with finite products.
- Any ring is a category with finite products.
- ...

First insight of Lawvere theories

Observation

Any model of an algebraic theory `is' a category with finite products.

Example

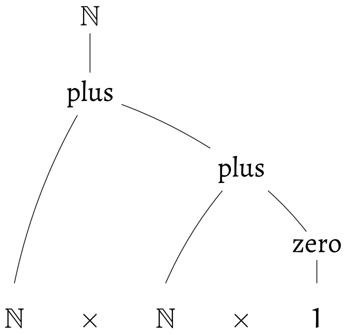
Category $\mathcal{C}_{\mathbb{N}}$ for the monoid of natural numbers and $+$:

- objects are finite `powers' of $\mathbb{N}$, e.g., $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$;
- morphisms are functions generated by
  - addition $\mathbb{N}\times\mathbb{N} \to \mathbb{N}$,
  - zero $1 \to \mathbb{N}$ (the map picking 0),
  - identities and composition,
  - pairing $\langle f_1, \ldots, f_n\rangle : \mathbb{N}^p \to \mathbb{N}^n$,
  - projections $\pi_{n,i} : \mathbb{N}^n \to \mathbb{N}$,
  - the unique map $\mathbb{N}^p \to 1$.

This is a subcategory of Set.

The category $\mathcal{C}_{\mathbb{N}}$ (looking closer)

- Any morphism $f : \mathbb{N}^p \to \mathbb{N}^n$ decomposes as $\langle f_1, \ldots, f_n \rangle$.
- Example morphism $\text{plus} \circ \langle \pi, \text{plus} \circ \langle \pi', (\text{zero} \circ !(\mathbb{N}^2)) \rangle \rangle : \mathbb{N}^2 \to \mathbb{N}$.
- A.k.a. $\text{plus}\,(x\,, (\text{plus}\,(y\,, \text{zero})))$.
- A.k.a.



$\approx$ circuits with sharing restricted to inputs.
- Variables: dealt with by projections.

Any monoid X 'is' a category $\mathcal{C}_X$ with finite products

General construction

Category $\mathcal{C}_X$ for the monoid (X, m, e):

- objects are finite 'powers' of X, e.g., $X \times X \times X$;
- morphisms are functions generated by
  - $m : X \times X \to X$,
  - $e : 1 \to X$ (the map picking e),
  - identities and composition,
  - pairing $\langle f_1, \ldots, f_n \rangle : X^p \to X^n$,
  - projections $\pi_{n,i} : X^n \to X$,
  - the unique map $X^p \to 1$.

Again a subcategory of Set.

The Lawvere theory for monoids

Definition

The category $\mathcal{L}_{\mathrm{monoid}}$ defined by:

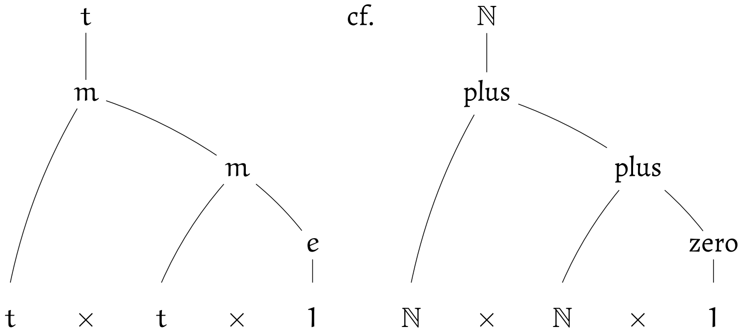
- Objects: one object, say t, and its formal finite powers $t \times \ldots \times t$.
- Morphisms $t \times \ldots \times t \to t$: terms generated by
  - binary m,
  - constant e,
  - in n (ordered) variables,

  up to a few equations.
- Morphisms $t^n \to t^p$: p-tuples of terms.
- Composition = simultaneous substitution.

Not directly a subcategory of sets.

The Lawvere theory for monoids

In $\mathcal{L}_{\text{monoid}}$, example morphism



Observation

There seems to be a `map' $\mathcal{L}_{\text{monoid}} \to \mathcal{C}_{\mathbb{N}}$.

Generalised monoids

Definition

A generalised monoid is

- a category $\mathcal{C}$ with finite products,
- an object $X \in \mathcal{C}$,
- morphisms $\mathrm{comp} : X \times X \to X$ and $\mathrm{unit} : 1 \to X$,
- satisfying the obvious associativity and unitality equations.

Example

- $\mathcal{C}_{\mathbb{N}}$,
- $\mathcal{C}_X$, for any monoid X,
- $\mathcal{L}_{\mathrm{monoid}}$.
- A bigger one: Set with, e.g., $\mathbb{N}$ and addition.

Morphisms

Definition

A morphism of generalised monoids is

- a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between underlying categories,

Wait wait, what's a functor?

Functors

Definition

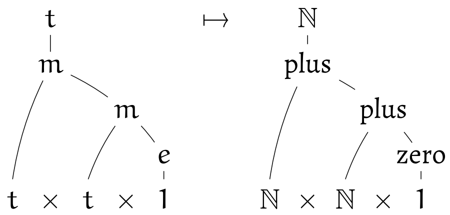
A functor $\mathcal{C} \to \mathcal{D}$ is a `morphism of categories':

- a map from objects of $\mathcal{C}$ to objects of $\mathcal{D}$,
- a map between morphisms (preserving source and target),

preserving composition and identities.

Example

The `map' $\mathcal{L}_{\text{monoid}} \to \mathcal{C}_{\mathbb{N}}$ determined by



Morphisms

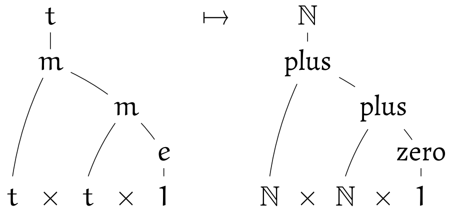
Definition

A morphism of generalised monoids is

- a functor $F : \mathcal{C} \to \mathcal{D}$ between underlying categories,
- preserving products: $F(A \times B) = F(A) \times F(B)$ (subtlety here, who can guess?),
- mapping X, comp, and unit in $\mathcal{C}$ to their counterparts in $\mathcal{D}$.

Example

Again, the functor $\mathcal{L}_{\text{monoid}} \to \mathcal{C}_{\mathbb{N}}$ determined by



In particular $t \times t \mapsto \mathbb{N} \times \mathbb{N}$.

The Lawvere theory for monoids

Proposition

Generalised monoids

$\simeq$

categories $\mathcal{C}$ with a finite-product preserving functor $\mathcal{L}_{\text{monoid}} \to \mathcal{C}$.

Intuitively, $\mathcal{L}_{\text{monoid}}$ serves as a definition of monoids.

Further example

What should be the Lawvere theory $\mathcal{L}_{\text{rings}}$ for rings?

- Objects: one object, say t, and its formal finite powers $t \times \ldots \times t$.
- Morphisms $t \times \ldots \times t \to t$: terms generated by
  - binary mult and add,
  - constants one and zero,
  - in n (ordered) variables,

  up to a few equations.
- Morphisms $t^n \to t^p$: p-tuples of terms.
- Composition = simultaneous substitution.

Lawvere theories: definition

Definition

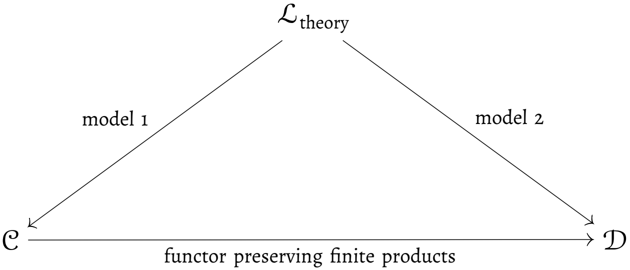
Lawvere theory: a category with

- finite products,
- objects formally generated by a set of `sorts'.

E.g., $t \times u \times t$, for sorts t and u.

What has been gained (quick summary)

- Signature + equations, i.e., theory $\mapsto$ category of models



  There are more general notions of morphisms...
- A notion of morphism between Lawvere theories: functors preserving finite-products; e.g.,

$$\mathcal{L}_{\text{monoid}} \hookrightarrow \mathcal{L}_{\text{ring}}.$$

What is missing?

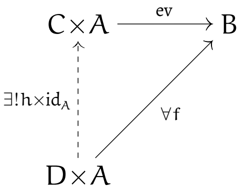
- Variable binding ($\lambda x\,.\,x \equiv \lambda x'.\,x'$).
- Dynamics $(\lambda x\,.\,M)N \rightsquigarrow M[x \mapsto N]$.

Need to generalise Lawvere theories!

Variable binding

We defined products by a property.

- Consider any objects $A, B \in \mathcal{C}$ with finite products;
  $C \times A \xrightarrow{ev} B$ is an exponential of A and B iff $\forall D \in \mathcal{C}$,



- Notation: $C = B^A$ and $h = \lambda f$.
- Intuition: B^A = function space, λf = currying of f.

Variable binding

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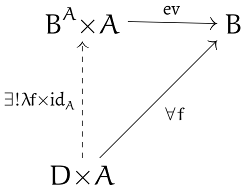

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Examples

- Set: B^A = set of functions $A \to B$.
- Gph, graphs: some convoluted construction of rare use (to my knowledge).
- Not Top! Have to restrict to compactly generated spaces.
- Scott domains. Particular posets, important in denotational semantics.

Cartesian closed categories = products + terminal object + exponentials

Definition

Cartesian closed category (CCC):

- a product $(A \times B, \pi, \pi')$ for all A, B,
- a terminal object 1,
- an exponential (B^A, ev) for all A, B.

Variable binding

Synopsis

Models of theories with binding `are' cartesian closed categories.

Example

The syntax for the pure λ-calculus yields a cartesian closed category $\mathcal{L}_\lambda$:

- objects are formal powers and exponentials of t, e.g., $t^t \times t$, $(t \times t \times t)^{t \times t^t}$,...
- morphisms are formally generated by
  - $\mathrm{lam} : t^t \to t$ and $\mathrm{app} : t \times t \to t$,
  - stuff needed for $\mathcal{L}_\lambda$ to be a CCC.

- Remark: $\mathcal{L}_\lambda$ contains more than the usual notion of syntax.
- Morphisms = simply-typed λ-terms up to βη-conversion.
- E.g., $\lambda x . M$ is here modelled as $\mathrm{lam}\,(\lambda x : t\, .\, [\![M]\!])$.

Dynamics

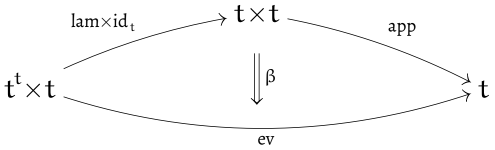
To model

$$(\lambda x.\, M)N \rightsquigarrow M[x \mapsto N]$$

we could add an equation

$$\mathrm{app}\,\langle \mathrm{lam}\,(\lambda x : t.\, M), N\rangle = (\lambda x : t.\, M)N.$$

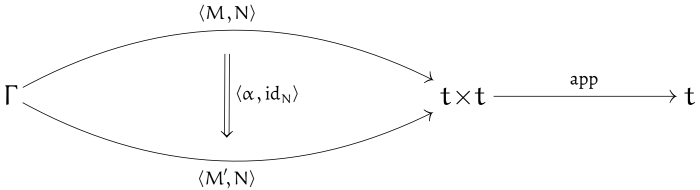
We prefer adding a 2-cell:



$\rightsquigarrow$ Cartesian closed 2-categories!

Example reduction

- In pure λ, if $M \rightsquigarrow M'$ then $M\,N \rightsquigarrow M'\,N$.
- Here, assuming $\alpha : M \Rightarrow N$, derive

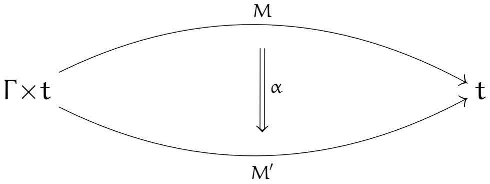


Syntactically

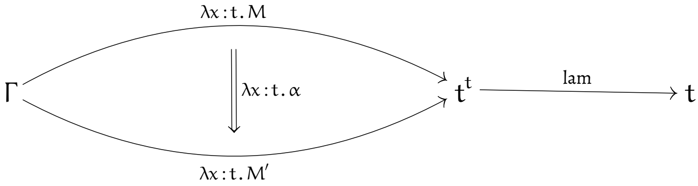
$\mathrm{app}\,\langle \alpha\,;\,\mathrm{id}_N \rangle : \mathrm{app}\,\langle M\,;\,N \rangle \Rightarrow \mathrm{app}\,\langle M'\,;\,N \rangle.$

Other example reduction

- In pure λ, if $M \rightsquigarrow M'$ then $\lambda x\,.\,M \rightsquigarrow \lambda x\,.\,M'$.
- Here, assuming



derive



Syntactically

$\lambda x : t\,.\,\alpha : \mathrm{lam}(\lambda x : t\,.\,M) \Rightarrow \mathrm{lam}(\lambda x : t\,.\,M').$

Proposition

$$\text{2-cells } M \Rightarrow N$$
$$\cong$$
reductions up to permutation equivalence (Lévy, late 70's; Bruggink, 2003).

Possible perspectives

- More involved examples.
- Dynamic properties of 2CCCs (following Hilken).
- Dynamic properties of morphisms.
- Formal links with other approaches.
- Extensions, e.g., dependent types.
- Coq library?